8. Determinant is in VBP

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Thm1 (Mahajan-Vinay '97) DET $\in$ VBP.

Def: Let $G=(V,E)$ be a directed weighted graph. A "clow" (acronym of "closed walk") in G is a walk $C=\langle w_1, \dots, w_\ell \rangle$ such that $w_1 = w_\ell$, and $w_1 < w_i$ for all $1 < i < \ell$ (with respect to a fixed order on the vertex set V).

$\text{Head}(C) = w_1$ is called the head of C.

A clow sequence in G is a sequence of clows, $W = \langle C_1, \dots, C_k \rangle$, such that (1) $\text{Head}(C_1) < \text{Head}(C_2) < \dots < \text{Head}(C_k)$, and (2) total number edges in W (counted with multiplicity) is $n := |V|$.

Remark: A (simple) cycle is a clow, and a cycle cover is a clow sequence.

For a cycle cover W, $\text{sgn}(W) = (-1)^{\#\text{ even cycles in } W}$.

It is easy to prove that $\text{sgn}(W) = (-1)^{n + \#\text{ cycles in } W}$, e.g. by decomposing into cycles.

For a clow sequence W, define $\text{sgn}(W) := (-1)^{n + \#\text{ clows in } W}$,

and the weight $w(W) := \prod_{e \in W} w(e)$ $\leftarrow$ viewed as a multiset. i.e. e may repeat.

Let A_G be the weighted adjacency matrix of G, i.e. $(A_G)_{i,j} = w(i,j)$.

Thm2: $$\det(A_G) = \sum_{W \text{ clow sequence in } G} \text{sgn}(W)\, w(W)$$

Pf: We claim:

Claim: $\exists$ an involution σ (i.e. $\sigma^2 = id$) on the set of clow sequences in G such that: If W is a cycle cover, then $\sigma(W) = W$.
If not, $\text{sgn}(\sigma(W)) = -\text{sgn}(W)$ and $w(\sigma(W)) = w(W)$.

Thm 2 follows from this claim and the fact $\det(A_G) = \sum_{W \text{ cycle cover}} \text{sgn}(W)\, w(W)$.

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So it remains to prove the claim.

Consider a clow sequence $W = \langle C_1, \dots, C_k \rangle$.

Choose the smallest i such that $\langle C_{i+1}, \dots, C_k \rangle$ is a set of disjoint simple cycles.

If $i = 0$, or equivalently, W is a cycle cover, let $\sigma(W) = W$.

Now suppose $i > 0$, i.e. W is not a cycle cover.

Traverse C_i starting from $\mathrm{Head}(C_i)$ until one of the following happens:

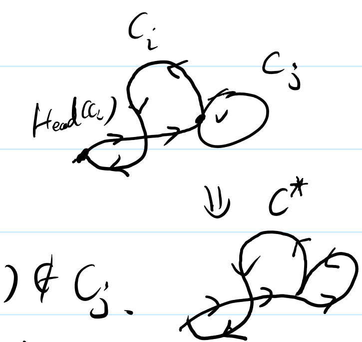
(1) We hit a vertex v that touches $C_j \in \{C_{i+1}, \dots, C_k\}$.

(2) We hit a vertex v that completes a simple cycle C within C_i.

As W is not a cycle cover, either (1) or (2) happens.

They cannot both happen. Otherwise, we hit v that completes a simple cycle and $v \in C_j$, but then (1) would happen earlier when we first visit v.

We define $\sigma(W)$ in cases (1) and (2):



Case (1): $v \in C_i$ touches $C_j \in \{C_{i+1}, \dots, C_k\}$

Merge C_i and C_j into C^*

Note: $\mathrm{Head}(C_i) < \mathrm{Head}(C_j)$ and hence $\mathrm{Head}(C_i) \notin C_j$.

So C^* is a clow with $\mathrm{Head}(C^*) = \mathrm{Head}(C_i)$.

Let $\sigma(W) = \langle C_1, \dots, C_{i-1}, C^*, C_{i+1}, \dots, C_{j-1}, C_{j+1}, \dots, C_k \rangle$.

Note $\mathrm{sgn}(\sigma(W)) = -\mathrm{sgn}(W)$ since # clows decreases by 1.

and $w(\sigma(W)) = w(W)$ since the edge multiset doesn't change.

Case (2): v completes a simple cycle C within C_i.

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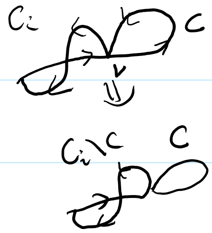

Say $C_i = \langle w_1, \cdots \underbrace{v, \cdots, v}_{C}, \cdots, w_\ell \rangle$

$\text{Head}(C_i) = w_1 \neq v$ by the definition of clows.

Decompose C_i into C and $C_i \setminus C$.

Then $\text{Head}(C_i \setminus C) = \text{Head}(C_i) < \text{Head}(C)$.

As we are not in Case (1), C is disjoint from $C_{i+1}, \cdots, C_k$

So $\text{Head}(C) \notin \{\text{Head}(C_{i+1}), \cdots, \text{Head}(C_k)\}$.

C inserted at the right position according to $\text{Head}(C)$

Let $\sigma(W) = \langle C_1, \cdots, C_{i-1}, C_i \setminus C, C_{i+1}, \cdots, C_k \rangle$

Note $\text{sgn}(\sigma(W)) = -\text{sgn}(W)$ since # clows increases by 1,

and $w(\sigma(W)) = w(W)$ since the edge multi-set doesn't change.

It is easy to verify that $\sigma^2 = \text{id}$ (and σ exchanges Cases (1) and (2))

$\square$

Proof of Thm 1 (sketch): Let G be the weighted directed graph with $w(i,j) = x_{ij}$ for $1 \leq i, j \leq n$.

We say $W = \langle C_1, \cdots, C_k \rangle$ is a partial clow sequence if $C_1, \cdots, C_{k-1}$ are clows, C_k is a prefix of a clow, and $\text{Head}(C_1) < \cdots < \text{Head}(C_k)$

For each
- p = parity of n + # clows already constructed
- h = head of the clow being constructed
- u = the vertex the current clow has reached
- i = # edge traversed in this and preceding clows,

let $f_{p,h,u,i}$ be the polynomial

$$\sum_{\substack{\text{partial clow} \\ \text{sequence } W \\ \text{in } G}} \prod_{\substack{e \in W \\ \text{multi-set}}} w(e) \in \mathbb{F}[x_{11}, \cdots, x_{nn}].$$

Using dynamic programming, build an ABP:

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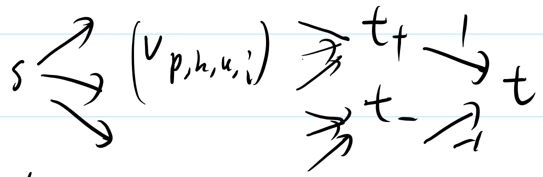

where for each (p,h,u,i), the ABP contains a vertex $v_{p,h,u,i}$ that computes $f_{p,h,u,i}$ (details left as an exercise. Or see Mahajan-Vinay)

Let t_+ (resp. t_-) collects the weights of all clow sequences with positive (resp. negative) sign.

And let $t = t_+ - t_-$. Then the ABP computes

$$\sum_{\substack{\text{clow sequence} \\ W \text{ in } G}} \operatorname{sgn}(W) \prod_{\substack{e \in W \\ \uparrow \text{multiset}}} w(e) = \det(A_G), \text{ where } A_G = (x_{ij})_{1 \le i,j \le n}.$$

$\square$

We say $t: \mathbb{N}^+ \to \mathbb{N}^+$ is qp-bounded if $\exists c > 0$ s.t. $t(n) \le 2^{(\log n)^c}$ for all $n \in \mathbb{N}^+$.

<u>Def</u>: VQP = set of polynomial families (f_n) s.t. f_n is computed by a circuit of size $\le t(n)$, where t is qp-bounded.

We say $(f_n) \le_{qp} (g_n)$ if $f_n \le_p g_{t(n)}$ for all n and a qp-bounded function t.

$\uparrow$ qp-projection

<u>Thm</u>: DET is complete in VP and VQP under qp-projections.

<u>Pf</u>: We know $\mathrm{DET} \in \mathrm{VP} \subseteq \mathrm{VQP}$.

Let $(f_n) \in \mathrm{VQP}$. Then f_n is computed by a circuit C of size $\le 2^{(\log n)^c}$, $c > 0$.

By depth reduction, we may assume $\operatorname{depth}(C) \le (\log(\operatorname{size}(C)))^2$

$\le (\log n)^{2c}$

Then f_n is computed by a formula of size $\le 2^{(\log n)^{2c}}$.

and hence an ABP of size $\le 2^{O((\log n)^{2c})}$

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As DET is ABP-complete under p-projections,

$f_n \leq \mathrm{DET}_m$, where $m \leq 2^{(\log n)^{c'}}$ for some $c' > 0$.

$\square$